Quantum effects in classical systems having complex energy

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Abstract
On the basis of extensive numerical studies it is argued that there are strong analogies between the probabilistic behavior of quantum systems defined by Hermitian Hamiltonians and the deterministic behavior of classical mechanical systems extended into the complex domain. Three models are examined: the quartic double-well potential $V(x) = x^4 - 5x^2$, the cubic potential $V(x) = \frac{1}{2}x^2 - gx^3$, and the periodic potential $V(x) = -\cos x$. For the quartic potential a wave packet that is initially localized in one side of the double-well can tunnel to the other side. Complex solutions to the classical equations of motion exhibit a remarkably analogous behavior. Furthermore, classical solutions come in two varieties, which resemble the even-parity and odd-parity quantum-mechanical bound states. For the cubic potential, a quantum wave packet that is initially in the quadratic portion of the potential near the origin will tunnel through the barrier and give rise to a probability current that flows out to infinity. The complex solutions to the corresponding classical equations of motion exhibit strongly analogous behavior. For the periodic potential a quantum particle whose energy lies between -1 and 1 can tunnel repeatedly between adjacent classically allowed regions and thus execute a localized random walk as it hops from region to region. Moreover, if the energy of the quantum particle lies in a conduction band, then the particle delocalizes and drifts freely through the periodic potential. A classical particle having complex energy executes a qualitatively analogous local random walk, and there exists a narrow energy band for which the classical particle becomes delocalized and moves freely through the potential.

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(Some figures in this article are in colour only in the electronic version)

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1. Introduction

Quantum mechanics and classical mechanics provide profoundly different descriptions of the physical world. In one-dimensional classical mechanics one is given a Hamiltonian of the form $H(x, p) = \frac{1}{2}p^2 + V(x)$. The motion of a particle modeled by this Hamiltonian is deterministic and is described by Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p} = p, \qquad \dot{p} = -\frac{\partial H}{\partial x} = -V'(x), \tag{1}$$

or equivalently, by Newton's law $-V'(x) = \ddot{x}$. The position $x(t)$ of a particle at time t is found by solving a *local* initial-value problem for these differential equations. The energy E of a particle, that is, the numerical value of the Hamiltonian, is a constant of the motion and can take on continuous values. Particle motion is restricted to the classically allowed regions, which are defined by $E \geqslant V(x)$. Because a particle may not enter a classically forbidden region, where $E < V(x)$, a classical particle may not travel between disconnected classically allowed regions.

In quantum mechanics Heisenberg's operator equations of motion and the time-dependent Schrödinger equation are posed as initial-value problems, just as Hamilton's equations are treated as initial-value problems in classical mechanics. However, the time-dependent Schrödinger equation is required to satisfy *nonlocal* boundary conditions that guarantee that the total probability of finding the particle is finite. For stationary states these boundary conditions demand that the eigenfunctions of the Hamiltonian satisfy a nonlocal boundary-value problem. As a consequence, for a rising potential that confines a classical particle the energy spectrum is discrete.

While classical mechanics consists of nothing more than solving a differential equation to find the exact trajectory of a particle, quantum mechanics is an abstract theory in which the physical state of the system is represented by a vector in a Hilbert space and predictions are probabilistic. The nonlocality mentioned above implies that physical measurements are subtle and difficult to perform. Quantum effects such as discretized energies, tunneling, and interference are a consequence of the nonlocal nature of the theory and are not intuitive. For example, one cannot speak of an actual path that a particle follows when it tunnels from one classically allowed region to another.

Complex-variable theory is of great assistance in providing an understanding of nonintuitive real-variable phenomena. It explains, for example, why the Taylor series for a real function $f(x)$ may cease to converge at a real value of x where $f(x)$ is smooth. (Series convergence is linked to the presence of singularities that may lie in the complex plane and not on the real axis.) Moreover, complex analysis shows the fundamental theorem of algebra, which is a deep property of the roots of polynomials, to be nothing more than a straightforward application of Liouville's theorem.

The objective of this conjectural paper is to demystify some well-known quantum effects by showing that their qualitative features can be reproduced very simply by the deterministic equations of classical mechanics (Newton's law) when these equations are extended to and solved in the complex plane. Specifically, we take the uncertainty principle $\Delta E \Delta t \gtrsim \frac{1}{2}\hbar$ to mean that there is intrinsic uncertainty in the energy of a particle, and in this paper we consider the possibility that this uncertainty may have an imaginary as well as a real part. We find that a deterministic classical particle whose energy has a small imaginary component can exhibit phenomena that are associated exclusively with quantum mechanics. We do not necessarily claim that quantum mechanics is a deterministic hidden-complex-variable theory. Indeed, there are important quantum phenomena, such as interference effects, that we cannot as yet

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reproduce by using complex classical mechanics[4]. However, the results that we obtain from complex classical mechanics bear a striking qualitative and quantitative resemblance to many well-known quantum phenomena such as tunneling and wave-like effects.

This paper is organized as follows: In section 2 we illustrate the power of complex analysis by using it to explain the quantization of energy. We show that in the complex domain the energy levels cease to be discrete and energy quantization can be explained topologically as the counting of sheets of a Riemann surface. In section 3 we describe the general features of classical particle trajectories when the energy of the classical particle is allowed to be complex. Specifically, classical trajectories that are closed and periodic when the energy is real cease to be closed when the energy becomes complex. In sections 4, 5 and 6 we examine the complex particle trajectories for three potentials whose quantum properties are well studied: the quartic double-well potential $x^4 - 5x^2$, the cubic potential $x^2 - gx^3$, and the periodic potential $-\cos x$. In each of these cases, we find that the corresponding complex classical system is able to mimic the quantum phenomena of tunneling, bound states of distinct parity, conduction bands, and energy gaps. Section 7 contains some concluding remarks.

2. Quantization from the complex-variable perspective

The notion of quantized energy levels is a central feature of quantum mechanics and is a dramatic departure from the continuous energy associated with classical mechanics. One can gain a different perspective on quantization if one extends quantum theory into the complex domain. To illustrate this, we consider a simple two-dimensional quantum system whose Hamiltonian is

$$H = H_0 + \epsilon H_I, \tag{2}$$

where the coupling constant ϵ is real. The diagonal matrix

$$H_0 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \tag{3}$$

whose two energy levels are a and b, describes the unperturbed system. The interaction is represented by the Hermitian off-diagonal matrix

$$H_I = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}. \tag{4}$$

The energy levels of H are evidently real and discrete:

$$E_\pm = \tfrac{1}{2}\left[a + b \pm \sqrt{(a-b)^2 + 4\epsilon^2 c^2}\right]. \tag{5}$$

An elementary way to understand the discreteness of these energy levels is to extend the coupling constant ϵ into the complex domain: Define the *energy function* $E(\epsilon)$ by

$$E(\epsilon) \equiv \tfrac{1}{2}\left[a + b + \sqrt{(a-b)^2 + 4\epsilon^2 c^2}\right]. \tag{6}$$

As a function of complex ϵ, $E(\epsilon)$ is double-valued, so it must be defined on a two-sheeted Riemann surface. These sheets are joined at a branch cut that connects the square-root branch points located at $\epsilon = \pm i(a - b)/(2c)$ (see figure 1). On the real-ϵ axis of the first sheet $E(\epsilon) = E_+$, and on the real-ϵ axis of the second sheet $E(\epsilon) = E_-$. On the Riemann surface the energy function $E(\epsilon)$ is smooth and *continuous* and is not a quantized function of complex ϵ. Indeed, along a continuous path that runs from a point ϵ_0 on the real axis on the first

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[4] There have been several alternative approaches to simulating quantum phenomena using complex-variable methods. For example, Goldfarb, Degani, Schiff, and Tannor [1] used complex actions in a framework of Bohmian mechanics and Yang [2] studied the complex Hamilton–Jacobi equations.

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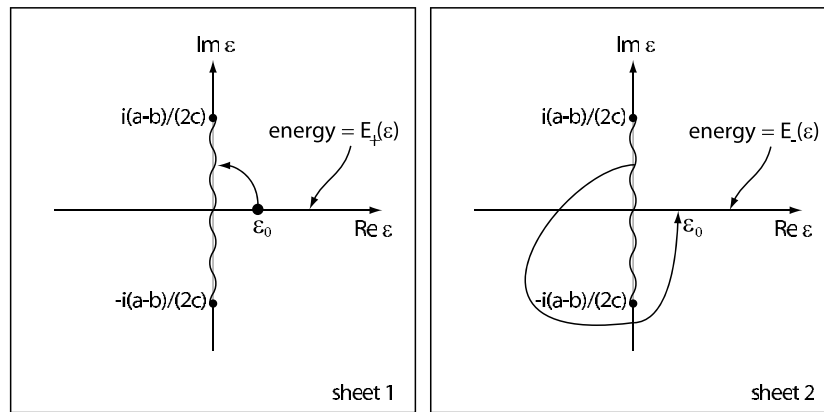
Figure 1. Two-sheeted Riemann surface for the energy function $E(\epsilon)$ in (6). On the Riemann surface this function is smooth and continuous, and the quantization of energy levels corresponds to counting the sheets on the surface. The energy appears to be discrete and quantized only if we limit the Riemann surface to the real-ϵ axis. A path from ϵ_0 on the real axis of sheet 1 to the corresponding point on the real axis of sheet 2 is shown. Along this path the energy eigenvalue E_+ continuously deforms to the other energy eigenvalue E_-.

sheet, crosses the branch cut, and goes to the corresponding point ϵ_0 on the real axis on the second sheet, the energy eigenvalue E_+ continuously deforms to E_-. (Such a path is shown on figure 1.) Thus, we see that the quantization in (5) is a consequence of the topological discreteness of the sheets that make up the Riemann surface. The energy function $E(\epsilon)$ appears quantized only if we restrict its domain to the real axes on the sheets of the Riemann surface.

To summarize, we have extended the Hamiltonian in (2) into the complex domain by complexifying the coupling constant ϵ and have obtained a clearer and deeper understanding of the nature of quantization. The topological picture of quantization described here is quite general, and it applies to more complicated systems, such as $H = p^2 + V(x) + \epsilon W(x)$, which have an infinite number of energy levels [3].

In general, the advantage of analyzing a system in the complex plane is that special features (like the discreteness of eigenvalues), which only occur on the real axis or which only emerge when we limit our attention to the real domain, can be seen to be part of a simpler and more general framework. In the rest of this paper we will examine complexified classical mechanics. We will see that while classical trajectories tend to be closed and periodic when the energy is strictly real, this special feature disappears and trajectories cease to be closed when the energy is allowed to be complex. Open trajectories are generic and, unlike closed trajectories, their behavior is rich and elaborate and bears a strong resemblance to some of the features that are thought to be restricted to the domain of quantum mechanics.

3. Classical mechanics in the complex domain

Given a classical Hamiltonian $H(x, p)$, the path $x(t)$ of a particle is fully determined by Hamilton's equations of motion (1) together with the initial conditions $x(0)$ and $p(0)$. The energy E is fixed by these initial conditions and is left invariant under the action of Hamilton's equations of motion. In elementary texts on classical mechanics the initial conditions are taken to be real so that the energy E is real and, in addition, particle trajectories are restricted to the real-x axis. However, in recent papers on $\mathcal{PT}$-symmetric classical mechanics, it has been

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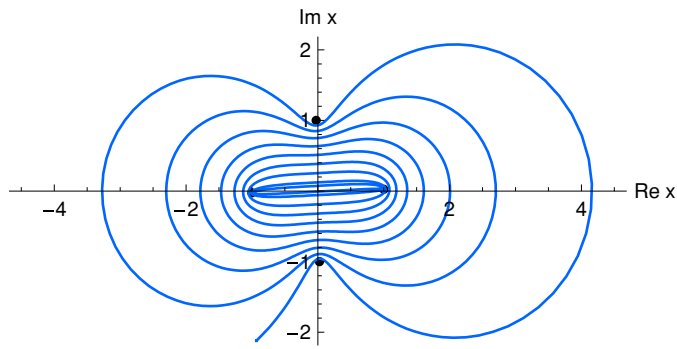
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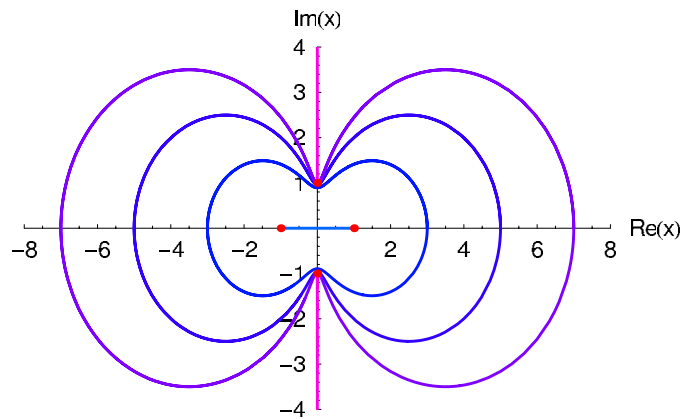
Figure 2. Classical trajectories in the complex-x plane representing the possible motions of a particle of energy $E = 1$. This motion is governed by the anharmonic-oscillator Hamiltonian $H = \frac{1}{2}p^2 + x^4$. There is one real trajectory that oscillates between the turning points at $x = \pm 1$ and an infinite family of nested complex trajectories that enclose the real turning points but lie inside the imaginary turning points at $\pm i$. (The turning points are indicated by dots.) Two other trajectories begin at the imaginary turning points and drift off to infinity along the imaginary-x axis. Apart from the trajectories beginning at $\pm i$, all trajectories are closed and periodic. All orbits in this figure have the same period $\sqrt{\pi/2}\,\Gamma\left(\frac{1}{4}\right)\big/\Gamma\left(\frac{3}{4}\right) = 3.708\,15\ldots$.

shown that it is interesting to study the complex as well as the real trajectories for systems having real energy [4–14].

We illustrate the real-energy trajectories of a classical–mechanical system by using the anharmonic oscillator, whose Hamiltonian is $H = \frac{1}{2}p^2 + x^4$. The classical trajectories for the energy $E = 1$ are shown in figure 2. There are four turning points located at $x = \pm 1, \pm i$ and indicated by dots. The so-called 'classically allowed' region is the portion of the real axis between $x = -1$ and $x = 1$, and a classical particle that is initially on this line segment will move parallel to the real axis and oscillate between the real turning points. The so-called 'classically forbidden' regions are the portions of the real-x axis for which $|x| > 1$, and a particle whose initial position is in one of these regions will have an initial motion that is perpendicular to the real axis. The particle will then enter the complex plane, make a sharp turn about the imaginary turning points, and return to its initial position. By virtue of Cauchy's theorem, all closed orbits in this figure have the same period $\sqrt{\pi/2}\,\Gamma\left(\frac{1}{4}\right)\big/\Gamma\left(\frac{3}{4}\right) = 3.708\,15\ldots$. There are two open orbits that run along the imaginary axis from i to $+i\infty$ and from $-i$ to $-i\infty$ in half this time. Note that two different classical trajectories can never cross.

The crucial feature illustrated in figure 2 is that all of the classical trajectories (except the two that run off to infinity along the imaginary axis) are closed. This means that we can view the system as a sort of complex atom. Because the classical orbits are closed, we can quantize the system and calculate the allowed real energies E_n by using the Bohr–Sommerfeld quantization formula $\oint \mathrm{d}x\, p = \left(n + \frac{1}{2}\right)\pi$ along *any* of these closed orbits to obtain the real discrete energy levels of the quantum anharmonic oscillator.

Recall that the Heisenberg uncertainty principle states that there is an intrinsic uncertainty in the energy E. Let us see what happens if this uncertainty in energy implies that it can take complex values. To begin with, since the turning points are determined by the value of the energy, they are slightly displaced from their positions in figure 2. However, the main effect is that while the classical trajectories still do not cross, they no longer need be closed and

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Figure 3. A single classical trajectory in the complex-x plane for a particle governed by the anharmonic-oscillator Hamiltonian $H = \frac{1}{2}p^2 + x^4$. This trajectory begins at $x = 1$ and represents the complex path of a particle whose energy $E = 1 + 0.1\mathrm{i}$ is complex. The trajectory is not periodic because it is not closed. The four turning points are indicated by dots. The trajectory does not cross itself.
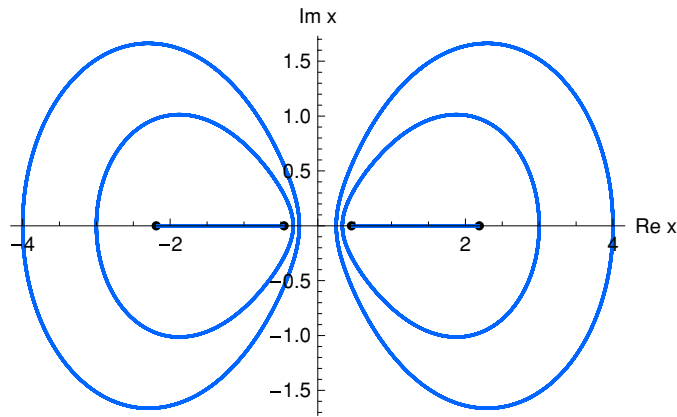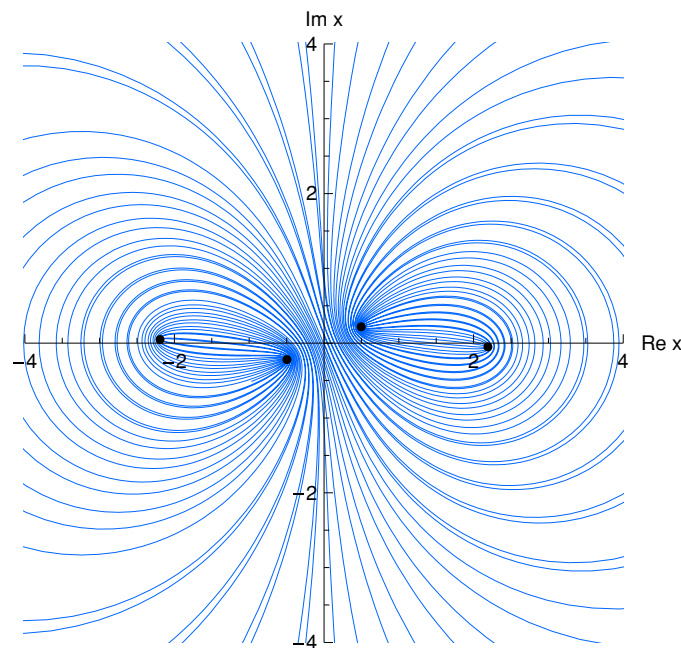

Figure 4. Six classical trajectories in the complex-x plane representing the motion of a particle of energy $E = -1$ in the potential $x^4 - 5x^2$. The turning points are located at $x = \pm 2.19$ and $x = \pm 0.46$ and are indicated by dots. Because the energy is real, the trajectories are all closed. The classical particle stays in either the right-half or the left-half plane and cannot cross the imaginary axis. Thus, when the energy is real, there is no effect analogous to tunneling.

periodic. In figure 3 a single trajectory for a particle whose energy is $E = 1 + 0.1\mathrm{i}$ is shown. The initial position is $x(0) = 1$ and the particle is allowed to travel for a time $t_{\max} = 35$.

4. Double-well potential

In this section we examine the double-well potential $x^4 - 5x^2$. A negative-energy quantum particle in such a potential tunnels back and forth between the classically allowed regions to the left and to the right of $x = 0$.

Let us first see what happens if we put a classical particle whose energy is real in such a potential well. We give an energy of $E = -1$ to this particle and plot some of the possible complex trajectories in figure 4. The turning points associated with this choice of energy lie on the real-x axis at $x = \pm 2.19$ and $x = \pm 0.46$ and are indicated by dots. Observe that the

Figure 5. Classical trajectory of a particle moving in the complex-x plane under the influence of a double-well $x^4 - 5x^2$ potential. The particle has complex energy $E = -1 - i$ and thus its trajectory does not close. The trajectory spirals inward around one pair of turning points and then spirals outward. The particle crosses the imaginary axis and spirals inward and then outward around the other pair of turning points. It then crosses the imaginary axis and repeats this behavior endlessly. At no point during this process does the trajectory cross itself. This classical-particle motion is analogous to behavior of a quantum particle that repeatedly tunnels back and forth between two classically allowed regions. Here, however, the particle does not disappear into the classically forbidden region during the tunneling process; rather, it moves along a well-defined path in the complex-x plane from one well to the other. Table 1 shows that this trajectory can be thought of as having odd parity.

classical trajectories are always confined to either the right-half or the left-half plane and do not cross the imaginary axis. Therefore, when the energy is real there is no effect analogous to quantum tunneling.

Next, we allow the energy of the classical particle to be complex: $E = -1 - i$. The open classical trajectories that result from such a complex energy are particularly interesting because their behavior is reminiscent of the phenomenon of quantum tunneling. Figure 5 shows a single trajectory that begins at $x = 0$. The particle moves into the right-half complex plane, and as time passes, the trajectory spirals inward around the right pair of turning points. The shape of the spiral is similar to that in figure 3. After many turns, the particle crosses the real axis between the two turning points and then begins to spiral outward. Eventually, the particle crosses the imaginary axis and begins to spiral inward around the left pair of turning points. The process then repeats: the particle eventually crosses the real axis, spirals outward, crosses the imaginary axis, and begins to spiral inward around the right pair of turning points.

This process of spiraling inward, spiraling outward, and crossing the imaginary axis continues endlessly, but at no point does the trajectory ever cross itself. During this process each pair of turning points acts like a strange attractor; the pair of turning points draws the trajectory inward along a spiral path, but then it drives the classical particle outward again

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Table 1. Imaginary-axis intercepts, directions and times for the classical trajectory shown in figure 5. Each time the trajectory crosses the imaginary axis we register the intercept in the first column, the direction of motion in the second column, and the time in the third column. This table becomes its mirror image under spatial reflection, so we classify the trajectory in figure 5 as having odd parity.

Imaginary crossing point	Direction	Time of crossing
0.909 592i	→	45.728 640
0.781 619i	←	5.366 490
0.441 760i	→	34.347 705
0.407 514i	←	16.764 145
0.253 436i	→	22.909 889
0.231 656i	←	28.205 183
0.118 499i	→	11.457 463
0.100 556i	←	39.658 755
0i	→	0
−0.017 057i	←	51.116 500
−0.082 877i	→	90.775 058
−0.136 772i	←	62.573 579
−0.210 728i	→	79.320 501
−0.276 231i	←	74.024 673
−0.376 304i	→	67.876 728
−0.479 881i	←	85.458 656
−0.690 666i	→	56.467 388
−1.121 155i	←	96.812 583

along another nested spiral path. The classical particle spends roughly half of its time spiraling under the influence of the left pair of turning points and the other half of its time spiraling under the influence of the right pair of turning points. The classical 'tunneling' process is less abstract and hence easier to understand than its quantum-mechanical analog. During quantum tunneling, the particle disappears from one classical region and reappears almost immediately in another classical region. We cannot ask which path the particle follows during this process. However, for a classical particle, it is clear how the particle travels from one classically allowed region to the other; it follows a well-defined path in the complex-x plane.

There is an even more surprising analogy between the quantum and classical systems. A stationary state of a quantum particle in a double well like $x^4 - 5x^2$ has a definite value of parity; that is, the eigenfunctions are either even or odd functions of x. The classical trajectory shown in figure 5 also exhibits a sort of parity, which we can observe if we keep track of the direction in which the particle is going when it crosses the imaginary axis. In table 1 we list in the first column the points at which the trajectory crosses the imaginary axis and in the second column we indicate whether the particle is crossing leftward or rightward. The third column indicates the time of the crossing. Under parity reflection the directions of the velocities in the second column reverse. However, the positions of the entries in the table must also be reflected about the central entry because under parity we replace the complex number x by $a - x$ for some value of a. Thus, this table becomes its mirror image under parity and we can classify the trajectory as having odd parity.

Let us now take a more negative value for the real part of the energy of the classical particle while keeping the imaginary part of the energy the same: $E = -2 - i$. The particle trajectory shown in figure 6 originates at $x = 0$ and has this energy. This trajectory is markedly different from that shown in figure 5 because the motion of the particle is confined to narrow bands or

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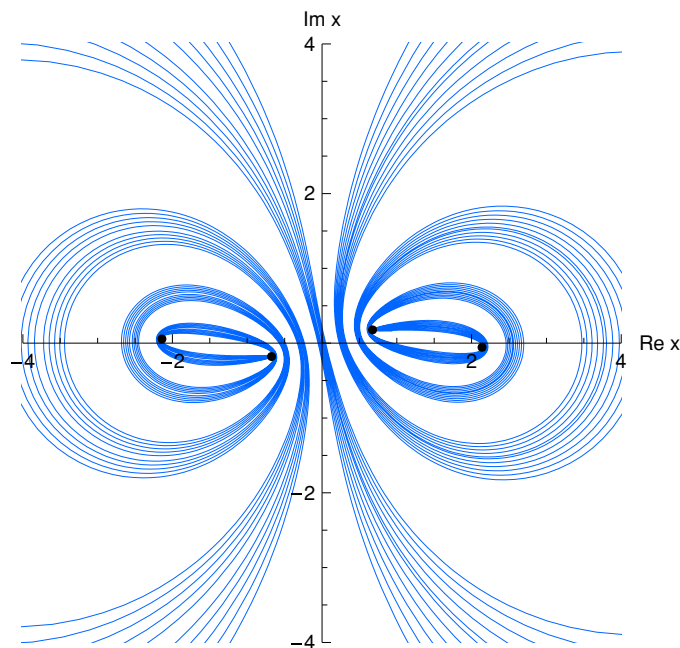
Figure 6. Classical trajectory in the complex-x plane for a particle in a double-well $x^4 - 5x^2$ potential. The particle begins its motion at the origin $x = 0$, but it has less real energy than the particle in figure 5: $E = -2 - i$. The trajectory is qualitatively different from that shown in figure 5 in that the path is confined to narrow ribbons and does not fill the complex-x plane. Also, because the real part of the energy of the particle is less than that for the particle in figure 5, the trajectory crosses the imaginary axis (leaps over the barrier) less frequently. Table 2 of imaginary-axis intercepts shows that we can interpret this trajectory as having even parity.

Table 2. Same as in table 1, but with entries corresponding to the classical trajectory in figure 6. There are fewer crossings than in table 1 because the classical particle has less real energy. In quantum-mechanical terms this means that the particle is less capable of leaping over the barrier between the wells.

Imaginary crossing point	Direction	Time of crossing
0.212 966i	←	85.604 840
0.114 590i	←	47.557 393
0.068 159i	←	28.534 298
0.407 514i	←	16.764 145
0.022 623i	←	9.511 410
0i	→	0
−0.045 318i	→	19.022 837
−0.091 223i	→	38.045 810
−0.187 424i	→	57.069 065
−0.210 728i	→	76.092 765
−0.239 354i	→	95.117 102

ribbons. More importantly, when we construct a crossing table for this figure (see table 2), we observe that the pattern of crossings is completely different from that in the second column of table 1. In this case the trajectory can be classified as having even parity.

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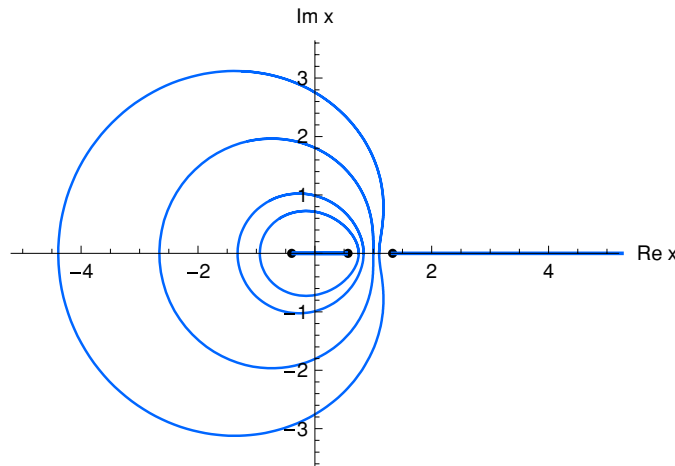
Figure 7. Six classical trajectories in the complex-x plane for a particle of energy $E = 0.1$ in the potential $\frac{1}{2}x^2 - \frac{1}{3}x^3$. The three turning points are located at $x = -0.398, 0.567, 1.331$. There are two classically allowed regions, one in which a classical particle oscillates between the turning points at -0.398 and 0.567, and a second that includes the real axis to the right of 1.331 in which a classical particle drifts off to infinity in finite time. All other classical trajectories are closed and periodic.

The parities of the quantum eigenfunctions associated with the double-well potential $x^4 - 5x^2$ alternate as the energy varies monotonically. Analogously, we have found that the parities of the classical trajectories also alternate as the real part of the energy changes monotonically.

5. Cubic potential

In this section we examine the cubic Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}x^2 - gx^3$, which is a model for a quantum particle in a long-lived metastable state. This particle is initially confined to the classically allowed region in the parabolic portion of the potential, but it eventually tunnels through the barrier and then moves rapidly off to $x = +\infty$. This Hamiltonian serves as an archetypal model for radioactive decay.

The energy levels of this Hamiltonian have been calculated for small g by using WKB theory [15], and the approximate formula for the ground-state energy is

$$E(g) \sim \frac{1}{2} - \frac{11}{8}g^2 - \mathrm{i}\,\frac{1}{g\sqrt{\pi}}\,\mathrm{e}^{-2/(15g^2)} \qquad (g \to 0^+). \tag{7}$$

The reciprocal of the imaginary part of the energy is an approximate measure of the lifetime τ of the metastable state:

$$\tau \approx g\sqrt{\pi}\,\mathrm{e}^{2/(15g^2)}. \tag{8}$$

Let us examine what happens to a classical particle under the influence of this potential. First, we choose $g = \frac{1}{3}$ and give this particle a *real* energy $E = 0.1$. Because this energy is real, the particle trajectories are closed and periodic (see figure 7). Periodic trajectories cannot represent a physical tunneling process in which a particle, initially confined in a potential well, gradually leaks out to infinity.

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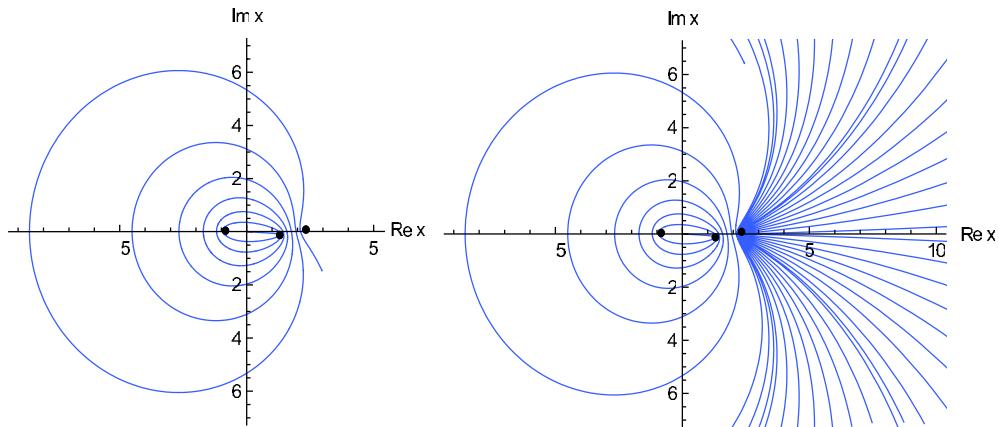
Figure 8. Trajectory in the complex-x plane of a classical particle of complex energy $E = 0.456 - 0.0489i$ in a $\frac{1}{2}x^2 - 2x^3/\sqrt{125}$ potential. The left trajectory begins at $x = 0$ and terminates at $t_{max} = 50$, while the right trajectory runs until $t_{max} = 200$. The right turning point takes control at about $t = 40$ and this is in good agreement with the lifetime τ of the quantum state, whose numerical value from (8) is about $\tau = 20$.

Next, we allow the energy of the particle to have an imaginary component. We take $g = 2/\sqrt{125}$ and set $E = 0.456 - 0.0489i$. This choice lifts the two outer turning points slightly above the real-x axis and pushes the central turning point below the axis, as shown in figure 8. The trajectory on the left represents a classical particle that starts at the origin and goes until $t_{max} = 50$, while the trajectory on the right runs until $t_{max} = 200$. Observe that the classical particle begins its motion by spiraling outward under the control of the two left turning points, which mark the edges of the confining region. After some time, the third turning point gradually takes control. We observe this change of influence as follows: initially, as the particle crosses the real axis to the right of the middle turning point, its trajectory is concave leftward, but as time passes, the trajectory becomes concave rightward. It is clear that by the fifth orbit the right turning point has gained control, and we can declare that the classical particle has now 'tunneled' out and escaped from the parabolic confining potential. The time at which this classical changeover occurs is approximately at $t = 40$. This is in good agreement with the lifetime of the quantum state in (8), whose numerical value is about 20.

6. Periodic potential

A periodic potential is used to model the behavior of a quantum particle in a crystal lattice. When the energy of such a particle lies below the top of the potential, there are infinitely many disconnected classically allowed regions on the real axis. Ordinarily, a quantum particle is confined to one such region, but has a finite probability of tunneling to an adjacent classically allowed region. Thus, such a particle hops at random from site to adjacent site in the crystal. However, for a narrow band (or bands) of energy the particle may drift freely from site to site in one direction. In such a *conduction band* the motion of the particle is said to be *delocalized*.

A classical particle in such a periodic potential exhibits these characteristic behaviors, but only if its energy is complex. If its energy is taken to be real, the particle merely exhibits periodic motion and remains confined forever to just one site in the lattice. Figure 9 illustrates this periodic motion for a particle of energy $E = -0.097\,54$ in a $-\cos x$ potential.

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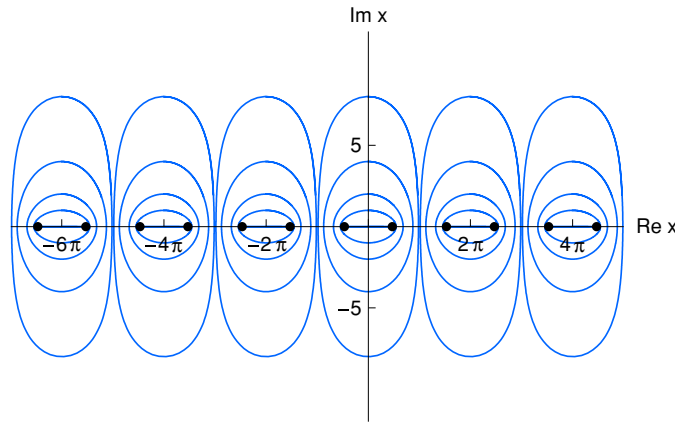
Figure 9. Classical trajectories in the complex-x plane for a particle of energy $E = -0.097\,54$ in a $-\cos x$ potential. The motion is periodic and the particle remains confined to a cell of width 2π. Five trajectories are shown for each cell. The trajectories shown here are the periodic analogs of the trajectories shown in figure 4.
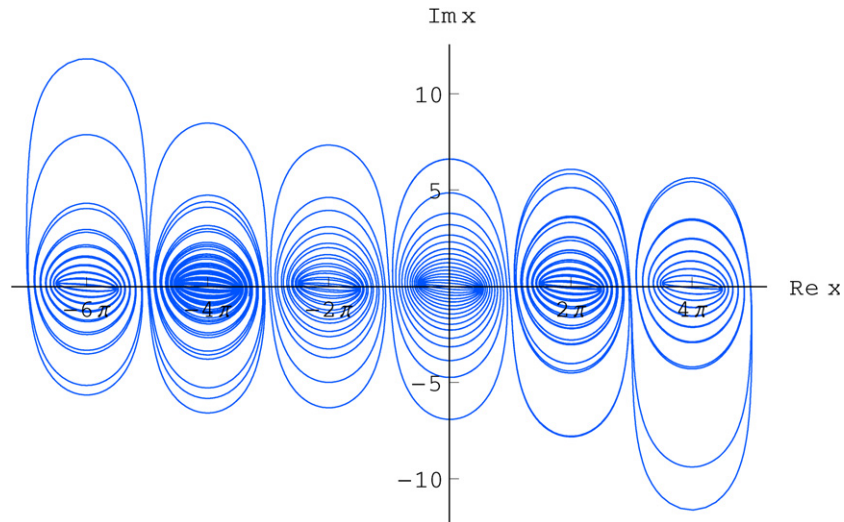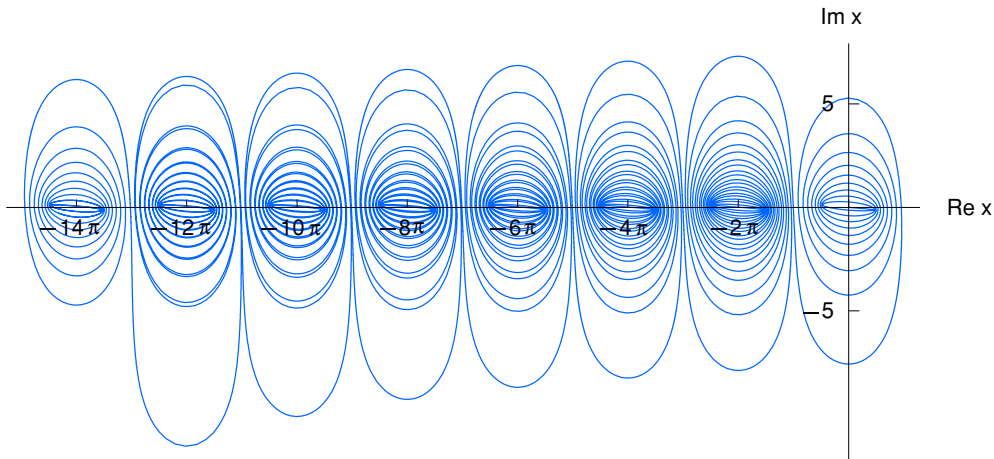

Figure 10. Classical trajectory in the complex-x plane of a particle of energy $E = -0.1 - 0.15\mathrm{i}$ in a $-\cos x$ potential. The particle starts at the origin $x = 0$, spirals outward, and hops to the adjacent site to the right. Then it spirals inward and back outward and hops to the right again. Then, it hops leftward five times, and back to the right once more. This deterministic 'random walk' is reminiscent of a localized quantum particle in a crystal.

For the same potential, if we take the energy to be complex $E = -0.1 - 0.15\mathrm{i}$, the classical particle now executes localized hopping from site to site (see figure 10). In this figure the particle starts at the origin $x = 0$ and 'tunnels' right, right, left, left, left, left, left, right, and so on, without ever crossing its trajectory. This deterministic 'random walk' is reminiscent of the behavior of a localized quantum particle hopping randomly from site to site in a crystal.

The tunneling rate of the classical particle depends on the imaginary part of its energy. To measure the time required for the particle in figure 10 to hop to an adjacent site, we simply

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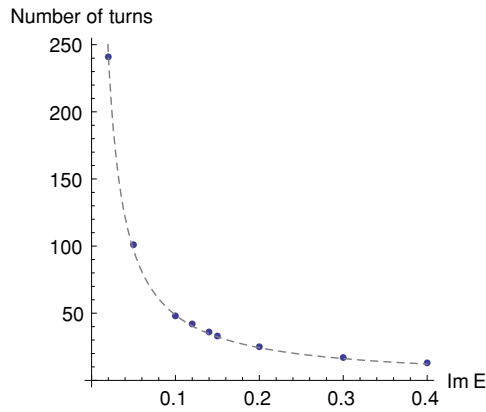
Figure 11. Number of turns in the spiral tunneling process shown in figure 10 that are required for a classical particle to hop from one site to an adjacent site versus the imaginary part of its energy. This graph shows that the product of the tunneling time and the imaginary part of the energy is a constant. For the time-energy uncertainty principle this product must be greater than $\frac{1}{2}$, and this product is about 17 for the classical $-\cos x$ potential.

Figure 12. Classical trajectory in the complex-x plane for a particle of energy $E = -0.097\,54 - 0.1278i$ in a $-\cos x$ potential. The particle starts at $x = 0$ and behaves like a delocalized quantum particle in a conduction band. It drifts leftward at a nearly constant rate, spiraling inward about 20 times and then outward about 20 more times before crossing to the next adjacent cell.

count the number of turns in the spiral path contained in each cell. This gives an extremely accurate measure of the time the particle spends in each cell before it hops to an adjacent cell. If we then vary the imaginary part of the energy and plot the relationship between Im E and the tunneling time, we obtain the graph shown in figure 11. We see from this graph that the product of Im E and the tunneling time is a constant. In units where $\hbar = 1$, for the time-energy quantum uncertainty principle this product should be greater than $\frac{1}{2}$, and this lower bound is saturated by the harmonic oscillator. Measured for the $-\cos x$ potential, we find that the numerical value of this product is approximately 17.

We have found that there is a narrow range for the real part of the energy in which the classical particle in the $-\cos x$ potential behaves as if it is a delocalized quantum particle in a conduction band; that is, the classical particle drifts consistently from site to site in the potential in one direction. A classical trajectory that illustrates this behavior is shown in figure 12. The energy of this particle is $E = -0.097\,54 - 0.1278i$. The band edges can be determined with great numerical precision; we find that when Im $E = -0.1278$, the range of real energy in this band is $-0.1008 < $ Re $E < -0.0971$.

7. Discussion

The classical differential equations that we have solved numerically in this paper can be solved exactly in terms of elliptic functions, but we have not done so because we are interested in their qualitative behavior only. Their solutions seem to suggest that many features thought to be only in the quantum arena can be reproduced in the context of complex classical mechanics. Of course, we have not shown that *all* quantum behavior can be recovered classically. In particular, we have not yet been able to observe the phenomenon of interference. (For example, we do not yet see the analog of the nodes in bound-state eigenfunctions.)

The ideas discussed here might be viewed as a vague alternative version of a hidden-variable formulation of quantum mechanics. The original idea of de Broglie, Bohm and Vigier was that a quantum system can be reduced to a deterministic system in which probabilities arise from the lack of knowledge concerning certain hidden variables. This approach encountered various difficulties, but an alternative way forward was suggested by Wiener and Della Riccia [16, 17], who argued that the hidden quantity in a coordinate-space representation of quantum mechanics is not the classical position of the particle but rather the momentum variable, which is integrated out and thus circumvents the issues associated with traditional approaches. In order to obtain the quantization condition, Wiener and Della Riccia introduced probability distributions over the classical phase space, thus obtaining the spectral resolution of the Liouville operator in terms of the eigenfunctions of the associated Schrödinger equation. The complex energy formulation of classical mechanics outlined here, if viewed as an alternative hidden variable theory, is close in spirit to that of Wiener and Della Riccia, but is distinct and more primitive in that we have not introduced probability distributions over the classical phase space. Nevertheless, the inaccessibility of the imaginary component of the energy in classical mechanics might necessitate introducing a probability distribution for the energy, which in turn might give rise to a more precise statement of uncertainty principles. The analogies between quantum mechanics and complex-energy classical mechanics reported here make further investigations worthwhile.

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